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TRANSVERSE VIBRATIONS OF HOLLOW THIN-WALLED CYLINDRICAL BEAMS[1]

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SUMMARY

The variational principle, differential equations, and boundary conditions considered appropriate to the analysis of transverse vibrations of hollow thin-walled cylindrical beams are shown. General solutions for the modes and frequencies of cantilever and free-free cylindrical beams of arbitrary cross section but of uniform thickness are given. The combined influence of the secondary effects of transverse shear deformation, shear lag, and longitudinal inertia is shown in the form of curves for cylinders of rectangular cross section and uniform thickness. The contribution of each of the secondary effects to the total reduction in the actual frequency is also indicated.

INTRODUCTION

The elementary theory of bending vibration is often inadequate for the accurate calculation of natural modes and frequencies of hollow, thin-walled cylindrical beams. Such secondary effects as transverse shear deformation, shear lag, and longitudinal inertia, which are not considered in the elementary theory of lateral oscillations, can have appreciable influence, particularly on the higher modes and frequencies of vibration. The effects of transverse shear deformation and of rotary (rather than longitudinal) inertia have been studied by many on the basis of the original investigations of Rayleigh (ref. 1) and Timoshenko (ref. 2). Anderson and Houbolt (ref. 3) have presented a procedure for including the effects of shear lag in the numerical calculation of modes and frequencies of box beams of rectangular cross section. However, there does not appear to exist a general solution for the vibration of hollow beams that incorporates the influence of all the secondary effects mentioned.

The purpose of the present report is threefold: First, to exhibit the variational principle, differential equations, and boundary conditions appropriate for the analysis of the uncoupled bending vibration of hollow thin-walled cylindrical beams; second, to give general solutions for cantilever and free-free cylinders of arbitrary cross section but of uniform thickness; and finally, to show quantitatively the influence of the secondary effects by means of numerical results for hollow beams of rectangular cross section of various lengths, widths, and depths.

SYMBOLS

A	cross-sectional area
A_n	Fourier coefficient
A_S	effective shear-carrying area
B_i	parameter defined in equation (30)
C	constant
E	modulus of elasticity
G	shear modulus of elasticity
I	moment of inertia
K	geometrical parameter defined in equation (29)
L	length of cantilever beam, half-length of free-free beam
N_i	parameter defined in equation (38)
T	maximum kinetic energy
U	maximum strain energy
a	half-depth of rectangular beam
b	half-width of rectangular beam
a_{mn}, b_n	Fourier series coefficients
i, j, m, n	integers
k_B	frequency coefficient, $\omega\sqrt{\frac{\mu L^4}{EI}}$
k_S	coefficient of shear rigidity, $\frac{1}{L}\sqrt{\frac{EI}{A_S G}}$
k_{RI}	coefficient of rotary inertia, $\frac{1}{L}\sqrt{\frac{I}{A}}$
p	perimeter of cross section
s	distance along periphery of cross section (see fig. 1)
t	wall thickness
$u(x,s)$	longitudinal displacement in x-direction
$w(x)$	vertical displacement in y-direction
x	longitudinal coordinate
y	vertical coordinate
$\bar{y}$	y-coordinate of center of gravity of cross section
γ_{xs}	shear strain

[1] Supersedes NACA TN 2682, "Transverse Vibrations of Hollow Thin-Walled Cylindrical Beams" by Bernard Budiansky and Edwin T. Kruszewski, 1952.

ϵ_x	longitudinal strain
θ	inclination of normal with vertical (see fig. 1)
λ	Lagrangian multiplier
μ	mass of beam per unit length
ρ	mass density of beam
σ	longitudinal direct stress
τ	shear stress
ω	natural frequency of beam
ω_0	natural frequency of beam calculated from elementary beam theory
δ_{ij}	Kronecker delta (1 if $i=j$; 0 if $i\neq j$)
φ	constraining relationship

BASIC EQUATIONS

Assumptions.—The problem to be considered is that of the natural bending vibration of a thin-walled hollow cylindrical beam whose cross section is symmetrical about at least one axis (see fig. 1). The transverse vibration is supposed to take place in the direction of this axis of symmetry of the cross section so that no torsional oscillations are induced.

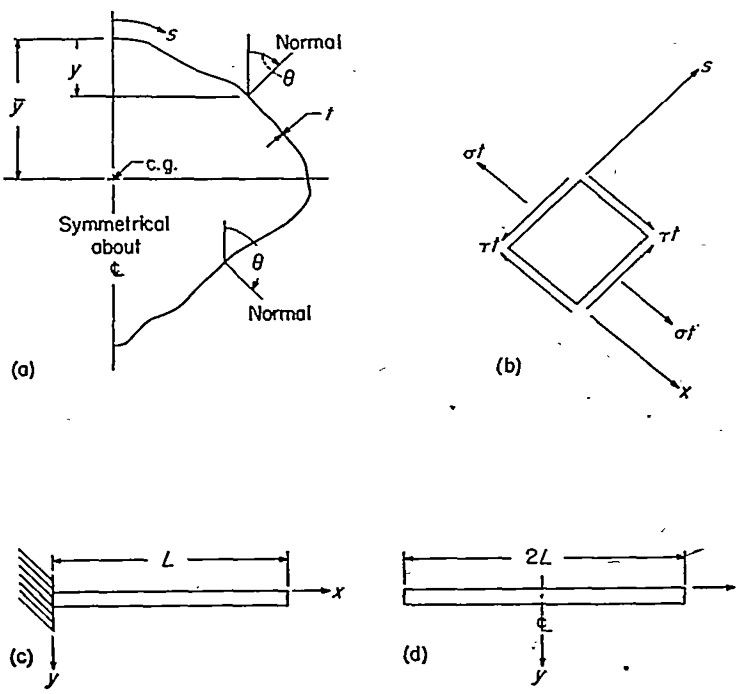
In the present analysis, the following simplifications are introduced:

(a) Changes in the size and shape of the cross section are neglected.

(b) Stress and strain are assumed to be uniform across the wall thickness.

(c) The small effect of circumferential stress upon longitudinal strain is neglected.

In accordance with statements (a) and (b), the distortions of the vibrating beam are completely described by the vertical displacement $w(x)$ of a cross section and the longitudinal displacement $u(x,s)$ of each point of the median line of the beam wall.



(a) Typical cross section. (b) Sign conventions.
(c). Cantilever beam. (d) Free-free beam.

FIGURE 1.—Coordinate systems and sign conventions.

The longitudinal and shear strains are given in terms of $u(x,s)$ and $w(x)$ as

$$\epsilon_x=\frac{\partial u}{\partial x} \tag{1}$$

and

$$\gamma_{xs}=\frac{\partial u}{\partial s}+\frac{dw}{dx}\sin\theta \tag{2}$$

and the corresponding stresses become

$$\sigma_x=E\frac{\partial u}{\partial x} \tag{3}$$

and

$$\tau_{xs}=G\left(\frac{\partial u}{\partial s}+\frac{dw}{dx}\sin\theta\right) \tag{4}$$

where θ is the inclination of the normal with the vertical (see fig. 1).

In elementary beam theory, where the effects of all shear distortion are neglected, the longitudinal distortion $u(x,s)$ is related to the vertical displacement $w(x)$ by

$$u\,(x,s)=(\bar{y}-y)\frac{dw}{dx}$$

where $\bar{y}$ is the y-coordinate of the center of gravity of the cross section. In the present report, however, $u(x,s)$ is allowed to be perfectly general, so that shear distortions (and consequently the so-called shear-lag and transverse-shear-deformation effects) are fully taken into account. Furthermore, because cross sections are not constrained to remain plane, the inertia effect associated with motion in the longitudinal direction is more properly designated as the effect of *longitudinal* inertia than the effect of *rotary* inertia.

Variational principle and geometrical boundary conditions.—The variational equation to be written is appropriate to beams whose ends are either fixed, simply supported, or free. For some such beam vibrating in a natural mode, the maximum strain energy is

$$U=\frac{1}{2}\int_0^L\oint E\left(\frac{\partial u}{\partial x}\right)^2 t\,ds\,dx+\frac{1}{2}\int_0^L\oint G\left(\frac{\partial u}{\partial s}+\frac{dw}{dx}\sin\theta\right)^2 t\,ds\,dx \tag{5}$$

where $u(x,s)$ and $w(x)$ are the amplitudes of displacement for the particular mode considered. The maximum kinetic energy is

$$T=\frac{1}{2}\int_0^L\oint \rho t\omega^2 w^2 ds\,dx+\frac{1}{2}\int_0^L\oint \rho t\omega^2 u^2 ds\,dx \tag{6}$$

where ω is the natural frequency of the mode under consideration and ρ is the mass density of the beam. The second term in equation (6) constitutes the contribution of longitudinal inertia to the kinetic energy.

A natural mode of vibration must satisfy the variational

equation

$$\delta(U-T)=0 \tag{7}$$

where the variation is taken independently with respect to $u(x,s)$ and $w(x)$ and with the provision that both $u(x,s)$ and $w(x)$ must satisfy the geometrical boundary conditions of the problem; furthermore, $u(x,s)$ must be periodic in the coordinate s with a period equal to the perimeter p. The geometrical boundary conditions are $w=0$ and $u=0$ at a fixed end and only $w=0$ at a simply supported end. At a free end no geometrical boundary conditions are imposed.

Differential equations and natural boundary conditions.— Equations (5), (6), and (7) in conjunction with the usual procedure of the calculus of variations yield the following simultaneous integrodifferential equations for u and w:

$$Et\frac{\partial^2 u}{\partial x^2}+\frac{\partial}{\partial s}\left[Gt\left(\frac{\partial u}{\partial s}+\frac{dw}{dx}\sin\theta\right)\right]+\rho t\omega^2 u=0 \tag{8}$$

$$\oint Gt\left(\frac{\partial^2 u}{\partial s\,\partial x}\sin\theta+\frac{d^2 w}{dx^2}\sin^2\theta\right)ds+\mu\omega^2 w=0 \tag{9}$$

where

$$\mu=\oint \rho t\,ds \tag{10}$$

and the boundary equations at each end of the beam are

$$\oint Et\left(\frac{\partial u}{\partial x}\right)\delta u\,ds=0 \tag{11}$$

$$\oint Gt\left(\frac{\partial u}{\partial s}+\frac{dw}{dx}\sin\theta\right)\sin\theta\,ds\,\delta w=0 \tag{12}$$

At a fixed end, both boundary equations (11) and (12) are satisfied by virtue of the fact that the geometrical boundary conditions require that both δu and δw be zero. At a simply supported end $\delta w=0$, but, since $\delta u(x,s)$ is perfectly arbitrary, the variational process forces the equality

$$Et\frac{\partial u}{\partial x}=0 \tag{13}$$

Finally, at a free end, since there are no geometrical constraints, both δu and δw are arbitrary and hence the variational process forces, in addition to equation (13), the equality

$$\oint Gt\left(\frac{\partial u}{\partial s}+\frac{dw}{dx}\sin\theta\right)\sin\theta\,ds=0 \tag{14}$$

Equations (13) and (14) constitute so-called "natural boundary conditions" because they are automatically satisfied as the result of a variational process. Equation (13) is recognized as the condition of zero longitudinal direct stress while equation (14) simply stipulates that the total vertical shear force vanish.

Thus to summarize, the appropriate boundary conditions required for the solution of equations (8) and (9) are

Fixed end:

$$w=0$$

$$u=0$$

Simply supported end:

$$w=0$$

$$Et\frac{\partial u}{\partial x}=0$$

Free end:

$$\oint Gt\left(\frac{\partial u}{\partial s}+\frac{dw}{dx}\sin\theta\right)\sin\theta\,ds=0$$

$$Et\frac{\partial u}{\partial x}=0$$

The integrodifferential equations (8) and (9), which specify equilibrium in the longitudinal and transverse directions respectively, can, of course, be written directly without recourse to the variational principle.

GENERAL SOLUTIONS FOR CYLINDERS OF UNIFORM WALL THICKNESS

The following exact solutions for cylinders of uniform wall thickness are carried out by means of Fourier series in conjunction with the application of the variational condition (eq. (7)). This procedure, which does not require explicit consideration of the natural boundary conditions, was believed to be more expedient than a direct attack upon the simultaneous integrodifferential equations (8) and (9) and all their associated boundary conditions.

Cantilever beam.—The geometrical boundary conditions, for a cantilever beam, as previously shown, are

$$w(0)=u(0,s)=0$$

(see fig. 1). Appropriate assumptions for the displacements $w(x)$ and $u(x,s)$ are

$$w(x)=C+\sum_{n=1,3,5}^{\infty} b_n\cos\frac{n\pi x}{2L} \tag{15}$$

and

$$u(x,s)\doteq\sum_{m=1,3,5}^{\infty}\sum_{n=0,1,2}^{\infty} a_{mn}\sin\frac{m\pi x}{2L}\cos\frac{2n\pi s}{p} \tag{16}$$

The condition $u(0,s)=0$ is satisfied by each term of equation (16); the condition

$$w(0)=C+\sum_{n=1,3,5}^{\infty} b_n=0 \tag{17}$$

is introduced into the variational procedure by means of the Lagrangian multiplier method. The choice of the particular trigonometric functions used in the Fourier series (15) and (16) was guided by consideration of the orthogonality required for the simplification of expressions in the strain energy. The constant C is needed in the expression for $w(x)$ in order that $w(L)$ be unrestricted.

Using equations (15) and (16) in equations (5) and (6) yields

$$U-T=$$

$$\frac{1}{2}\int_0^L\oint Et\left(\sum_{m=1,3,5}^{\infty}\sum_{n=0,1,2}^{\infty}a_{mn}\frac{m\pi}{2L}\cos\frac{m\pi x}{2L}\cos\frac{2n\pi s}{p}\right)^2 ds\,dx+$$

$$\frac{1}{2}\int_0^L\oint Gt\left(\sum_{m=1,3,5}^{\infty}\sum_{n=0,1,2}^{\infty}-a_{mn}\frac{2n\pi}{p}\sin\frac{m\pi x}{2L}\sin\frac{2n\pi s}{p}+\right.$$

$$\left.\sin\theta\sum_{n=1,3,5}^{\infty}-b_n\frac{n\pi}{2L}\sin\frac{n\pi x}{2L}\right)^2 ds\,dx-$$

$$\frac{1}{2}\int_0^L\oint\omega^2\rho t\left(\sum_{n=1,3,5}^{\infty}b_n\cos\frac{n\pi x}{2L}+C\right)^2 ds\,dx-$$

$$\frac{\omega^2}{2}\int_0^L\oint\rho t\left(\sum_{m=1,3,5}^{\infty}\sum_{n=0,1,2}^{\infty}a_{mn}\sin\frac{m\pi x}{2L}\cos\frac{2n\pi s}{p}\right)^2 ds\,dx \quad (18)$$

To make equation (18) stationary and at the same time satisfy the constraining relationship

$$\varphi=C+\sum_{n=1,3,5}^{\infty}b_n=0 \quad (19)$$

it is sufficient to set

$$\delta(U-T-\lambda\varphi)=0 \quad (20)$$

where the variation is with respect to the a's, b's, and C considered as independent variables; here λ is a Lagrangian multiplier. This variational process results in the following equations:

$$\frac{\partial(U-T)}{\partial b_i}-\lambda\frac{\partial\varphi}{\partial b_i}=\sum_{n=0,1,2}^{\infty}Gt\frac{in\pi^2}{2}\frac{A_n}{2}a_{in}+G\left(\frac{i\pi}{2L}\right)^2\frac{A_SL}{2}b_i-$$

$$\mu\omega^2\frac{L}{2}b_i-(-1)^{\frac{i-1}{2}}\frac{2L}{i\pi}\mu\omega^2C-\lambda$$

$$=0 \qquad (i=1,3,5,\ldots) \quad (21)$$

$$\frac{\partial(U-T)}{\partial a_{ij}}=Et\left(\frac{i\pi}{2L}\right)^2\frac{Lp}{4}(1+\delta_{0j})a_{ij}+Gt\left(\frac{2j\pi}{p}\right)^2\frac{Lp}{4}a_{ij}+$$

$$Gt\frac{ij\pi^2}{4}A_jb_i-\omega^2\rho t\frac{Lp}{4}(1+\delta_{0j})a_{ij}=0$$

$$(i=1,3,5,\ldots)$$
$$(j=0,1,2,\ldots) \quad (22)$$

$$\frac{\partial(U-T)}{\partial C}-\lambda\frac{\partial\varphi}{\partial C}=-\mu\omega^2\sum_{n=1,3,5}^{\infty}\frac{2L}{n\pi}(-1)^{\frac{n-1}{2}}b_n-\mu\omega^2LC-\lambda$$

$$=0 \quad (23)$$

where

$$A_n=\frac{2}{p}\oint\sin\theta\sin\frac{2n\pi s}{p}\,ds \quad (24)$$

$$A_S=\oint t\sin^2\theta\,ds \quad (25)$$

With the use of the nondimensional parameters

$$k_B^2=\frac{\mu L^4}{EI}\omega^2 \quad (26)$$

$$k_S^2=\frac{EI}{A_SGL^2} \quad (27)$$

$$k_{RI}^2=\frac{I}{ptL^2}=\frac{I}{AL^2} \quad (28)$$

$$K^2=\frac{16I}{A_Sp^2} \quad (29)$$

and

$$B_i^2=i^2-k_{RI}^2k_B^2\left(\frac{2}{\pi}\right)^2 \quad (30)$$

equations (21), (22), and (23) may be reduced to

$$\sum_{n=0,1,2}^{\infty}\frac{in\pi^2}{4k_S^2}\frac{Lt}{A_S}A_na_{in}+\frac{1}{2}\left(\frac{i\pi}{2}\right)^2\frac{1}{k_S^2}b_i-\frac{1}{2}k_B^2b_i-$$

$$(-1)^{\frac{i-1}{2}}\frac{2}{i\pi}k_B^2C-\frac{L^3\lambda}{EI}=0 \qquad (i=1,3,5,\ldots) \quad (31)$$

$$(k_S^2B_i^2+K^2j^2)(1+\delta_{0j})a_{ij}+K^2\frac{p}{4L}A_jijb_i=0$$
$$(i=1,3,5,\ldots)$$
$$(j=0,1,2,\ldots) \quad (32)$$

$$k_B^2\sum_{n=1,3,5}^{\infty}\frac{2}{n\pi}(-1)^{\frac{n-1}{2}}b_n+k_B^2C+\frac{L^3\lambda}{EI}=0 \quad (33)$$

For $j=0$, equation (32) becomes

$$k_S^2\left[i^2-k_B^2k_{RI}^2\left(\frac{2}{\pi}\right)^2\right]a_{i0}=0 \qquad (i=1,3,5,\ldots) \quad (34)$$

Equation (34) is not coupled to any of equations (31) to (33). A given value of a_{i0} corresponds to the amplitude of the ith mode of longitudinal oscillation, and if this value of a_{i0} is not equal to 0, then equation (34) simply gives the frequency of this longitudinal mode. Consequently those equations in equation (32) for values of $j=0$ are not associated with transverse bending and so are ignored henceforth. For the remaining values of j (that is, $j\neq0$) equation (32) yields

$$a_{ij}=\frac{-K^2\frac{p}{4L}A_jij}{k_S^2B_i^2+K^2j^2}b_i \qquad \begin{matrix}(i=1,3,5,\ldots)\\(j=1,2,3,\ldots)\end{matrix} \quad (35)$$

Substituting the expression for a_{ij} in equation (35) into equation (31) and solving for b_i gives

$$b_i=\frac{(-1)^{\frac{i-1}{2}}\frac{2}{i\pi}k_B^2C+\frac{L^3\lambda}{EI}}{N_i} \qquad (i=1,3,5,\ldots) \quad (36)$$

where

$$N_i=\frac{i^2\pi^2}{8}\frac{1}{k_S^2}-\sum_{n=1,2,3}^{\infty}\frac{\frac{\pi^2}{16}\frac{K^2}{k_S^2}\frac{A}{A_S}A_n^2(in)^2}{k_S^2B_i^2+K^2n^2}-\frac{1}{2}k_B^2 \tag{37}$$

In the appendix this expression for N_i is shown to be equivalent to

$$N_i=\frac{i^2\pi^4}{32}B_i^2-\frac{i^2\pi^2}{16}\frac{A}{A_S}B_i^4k_S^2\sum_{n=1,2,3}^{\infty}\frac{A_n^2}{K^2n^2(k_S^2B_i^2+K^2n^2)}-\frac{1}{2}k_B^2 \tag{38}$$

Since the series in equation (38) is considerably more quickly convergent than that in equation (37), equation (38) should be used in actual numerical calculations of N_i.

Substitution of equation (36) into equation (33) and the constraining-relationship equation (19) gives the following two homogeneous equations in C and λ:

$$k_B^2\left[1+k_B^2\sum_{n=1,3,5}^{\infty}\left(\frac{2}{n\pi}\right)^2\frac{1}{N_n}\right]C+\left[1+k_B^2\sum_{n=1,3,5}^{\infty}(-1)^{\frac{n-1}{2}}\frac{2}{n\pi}\frac{1}{N_n}\right]\frac{L^3\lambda}{EI}=0 \tag{39a}$$

$$\left[1+k_B^2\sum_{n=1,3,5}^{\infty}(-1)^{\frac{n-1}{2}}\frac{2}{n\pi}\frac{1}{N_n}\right]C+\left(\sum_{n=1,3,5}^{\infty}\frac{1}{N_n}\right)\frac{L^3\lambda}{EI}=0 \tag{39b}$$

Finally the condition for a nontrivial solution for C and λ gives the frequency equation

$$\begin{vmatrix} k_B^2\left[1+k_B^2\sum_{n=1,3,5}^{\infty}\left(\frac{2}{n\pi}\right)^2\frac{1}{N_n}\right] & 1+k_B^2\sum_{n=1,3,5}^{\infty}(-1)^{\frac{n-1}{2}}\frac{2}{n\pi}\frac{1}{N_n} \\ 1+k_B^2\sum_{n=1,3,5}^{\infty}(-1)^{\frac{n-1}{2}}\frac{2}{n\pi}\frac{1}{N_n} & \sum_{n=1,3,5}^{\infty}\frac{1}{N_n}\end{vmatrix}=0 \tag{40}$$

which the frequency parameter k_B must satisfy. Since the terms of the infinite series which appear in the frequency equation contain k_B itself, the roots of equation (40) are most conveniently found by trial. Fortunately the infinite series in equation (40) as well as the series in the definition of N_i converge rapidly so that only a few terms are needed to evaluate them with sufficient accuracy.

Once k_B has been determined for a particular mode, the corresponding mode shape can be found by letting $C=1$ and solving either of equations (39) for λ and then finally evaluating b_i and a_{ij} successively from equations (36) and (35).

Free-free beam—symmetrical modes.—If the origin of a free-free beam of length $2L$ is taken at the midspan (see fig. 1), the form of the Fourier series assumed for $w(x)$ and $u(x,s)$ when the beam is undergoing a symmetrical mode of vibration may be exactly the same as that assumed for the cantilever beam of length L (see eqs. (15) and (16)). The only difference in the ensuing calculations is that the constraining condition (19) is not introduced. Consequently, it can be readily seen that the frequency equation for the symmetrically vibrating free-free beam is obtained from equation (39a) by setting $\lambda=0$ and is

$$k_B^2\left[1+k_B^2\sum_{n=1,3,5}^{\infty}\left(\frac{2}{n\pi}\right)^2\frac{1}{N_n}\right]=0 \tag{41}$$

After a particular root k_B is found from equation (41), the shape of the corresponding symmetrical free-free mode may be obtained from equations (36) (with $\lambda=0$) and equations (35).

Free-free beam—antisymmetrical modes.—Consider a free-free beam of length $2L$ undergoing antisymmetrical vibrations. Explicit consideration need be given only to the right half of the beam (see fig. 1), and for this half-beam the only geometrical boundary condition that must be imposed is that $w(0)=0$. The spanwise displacement $u(0,s)$ is unrestrained by virtue of antisymmetry.

Appropriate assumptions for the displacements $w(x)$ and $u(x,s)$ are then

$$w(x)=\sum_{n=2,4,6}^{\infty}b_n\sin\frac{n\pi x}{2L}+Cx \tag{42}$$

and

$$u(x,s)=\sum_{m=0,2,4}^{\infty}\sum_{n=1,2,3}^{\infty} a_{mn}\cos\frac{m\pi x}{2L}\cos\frac{2n\pi s}{p} \tag{43}$$

The linear portion Cx of the expression for $w(x)$ is needed in order to give the beam sufficient freedom at the tip ($x=L$). The choice of the particular trigonometric function in the series expansion for $u(x,s)$ was, as in the case of the cantilever beam, guided by consideration of the orthogonality required for the simplification of the expressions in the strain energy. The zeroeth term in the series for $u(x,s)$ in the s-direction was omitted because it only leads to the frequency equation for longitudinal oscillations.

Using equations (42) and (43) in equations (5) and (6) yields

$$\begin{aligned} U-T=&\frac{1}{2}\int_0^L\oint Et\left(\sum_{m=0,2,4}^{\infty}\sum_{n=1,2,3}^{\infty}-a_{mn}\frac{m\pi}{2L}\sin\frac{m\pi x}{2L}\cos\frac{2n\pi s}{p}\right)^2 ds\,dx+\frac{1}{2}\int_0^L\oint Gt\Bigg[\sum_{m=0,2,4}^{\infty}\sum_{n=1,2,3}^{\infty}-a_{mn}\frac{2n\pi}{p}\cos\frac{m\pi x}{2L}\sin\frac{2n\pi s}{p}+\\ &\sin\theta\left(\sum_{n=2,4,6}^{\infty}b_n\frac{n\pi}{2L}\cos\frac{n\pi x}{2L}+C\right)\Bigg]^2 ds\,dx-\frac{1}{2}\int_0^L\oint \rho t\omega^2\left(\sum_{n=2,4,6}^{\infty}b_n\sin\frac{n\pi x}{2L}+Cx\right)^2 ds\,dx-\\ &\frac{1}{2}\int_0^L\oint \rho t\omega^2\left(\sum_{m=0,2,4}^{\infty}\sum_{n=1,2,3}^{\infty}a_{mn}\cos\frac{m\pi x}{2L}\cos\frac{2n\pi s}{p}\right)^2 ds\,dx \end{aligned} \tag{44}$$

The variation of equation (44) with respect to the a's, b's, and C gives, after suitable simplification,

$$(B_i^2k_S^2+K^2j^2)\,a_{ij}-K^2\frac{p}{4L}A_jijb_i=0 \quad \begin{matrix}(i=2,4,6,\ldots)\\(j=1,2,3,\ldots)\end{matrix} \tag{45}$$

$$\left[K^2j^2-k_S^2k_{RI}^2k_B^2\left(\frac{2}{\pi}\right)^2\right]a_{0j}-K^2\frac{p}{2\pi}A_jjC=0 \quad (j=1,2,3,\ldots) \tag{46}$$

$$\sum_{n=1,2,3}^{\infty}\frac{1}{k_S^2}\frac{Lt}{A_S}\frac{in\pi^2}{4}A_na_{in}-\frac{1}{2}\frac{1}{k_S^2}\left(\frac{i\pi}{2}\right)^2b_i+\frac{1}{2}k_B^2b_i-(-1)^{i/2}\frac{2}{i\pi}k_B^2CL=0 \quad (i=2,4,6,\ldots) \tag{47}$$

$$\sum_{n=1,2,3}^{\infty}\frac{1}{k_S^2}\frac{Lt}{A_S}n\pi A_na_{0n}-\frac{CL}{k_S^2}-k_B^2\sum_{n=2,4,6}^{\infty}\frac{2}{n\pi}(-1)^{n/2}b_n+\frac{1}{3}k_B^2CL=0 \tag{48}$$

From equation (45)

$$a_{ij}=\frac{K^2\frac{p}{4L}A_jij}{B_i^2k_S^2+K^2j^2}b_i \quad \begin{matrix}(i=2,4,6,\ldots)\\(j=1,2,3,\ldots)\end{matrix} \tag{49}$$

which, except for sign, is the same expression as that obtained for the cantilever and symmetrically vibrating free-free beams (eq. (35)). From equation (46)

$$a_{0j}=\frac{K^2\frac{p}{2\pi}A_jj}{B_0^2k_S^2+K^2j^2}C \quad (j=1,2,3,\ldots) \tag{50}$$

Substitution of equation (49) into equation (47) gives

$$b_i=-(-1)^{i/2}\frac{2}{i\pi}\frac{k_B^2}{N_i}CL \quad (i=2,4,6,\ldots) \tag{51}$$

where N_i is defined in equation (37).

Substitution of equations (50) and (51) into equation (48) and simplification gives as the frequency equation for the antisymmetrically vibrating free-free beam

$$k_B^2\left[\sum_{n=2,4,6}^{\infty}\left(\frac{2}{n\pi}\right)^2\frac{k_B^2}{N_n}+\frac{2}{\pi^2}\frac{A}{A_S}k_{RI}^2\sum_{n=1,2,3}^{\infty}\frac{A_n^2}{B_0^2k_S^2+K^2n^2}+\frac{1}{3}\right]=0 \tag{52}$$

After a particular value of k_B is found from equation (52), the shape of the corresponding antisymmetrical free-free mode may be obtained by giving C the arbitrary value of unity and calculating the b's and a's successively from equations (51), (50), and (49).

Discussion of parameters.—The parameters entering in the frequency equations merit discussion. The unknown natural frequency is contained only in the frequency coefficient k_B, which is defined by the formula $\omega=k_B\sqrt{\frac{EI}{\mu L^4}}$, and is in common use in beam-vibration analysis. The parameters k_S and k_{RI} are identical with the shear and inertia parameters defined in reference 4, which considers the effect of only transverse shear and rotary inertia on beam vibrations. The quantity A_S which appears in the present definition of k_S is actually the effective shear-carrying area when plane sections are constrained to remain plane; that is, when shear lag is neglected. The remaining parameters appearing in the present derivation, namely, A/A_S, K, and A_1, A_2, . . . are essentially shape parameters which actually depend only on the contour of the cross section; as shown in the appendix,

$$\frac{A_S}{A}=\frac{1}{2}\sum_{n=1,2,3}^{\infty}A_n^2$$

and

$$K^2=\frac{2}{\pi^2}\frac{A}{A_S}\sum_{n=1,2,3}^{\infty}\frac{A_n^2}{n^2}$$

and the A_n's are simply the Fourier coefficients of the function $\sin\theta$, which is dependent only on the shape of the

cross section. These shape parameters are related to shear-lag effects and their interaction with transverse shear and longitudinal inertia.

The effect of longitudinal inertia is associated with the parameter k_{RI}. If the effect of longitudinal inertia is to be neglected, it is sufficient to set k_{RI} equal to zero in the final frequency equation. If k_{RI} is equal to zero, B_t becomes independent of k_B. Appreciable simplification in a trial-and-error solution for the natural frequency then results since, with B_t independent of k_B, the infinite summation contained in N_t is also independent of k_B and need be calculated only once for any particular beam. As is shown in the following section, the effect of disregarding the influence of longitudinal inertia may often be negligible.

Without presentation of details, it may be mentioned that for the case of a circular cylinder, which has no shear lag, all the A_n's except A_1 vanish and the frequency equations (40), (41), and (52) may be put into closed forms identical to those given in reference 4. Again, if in the general frequency equations k_S is set equal to zero, the equations may be put into closed forms equivalent to those of reference 4 where only rotary inertia is considered.

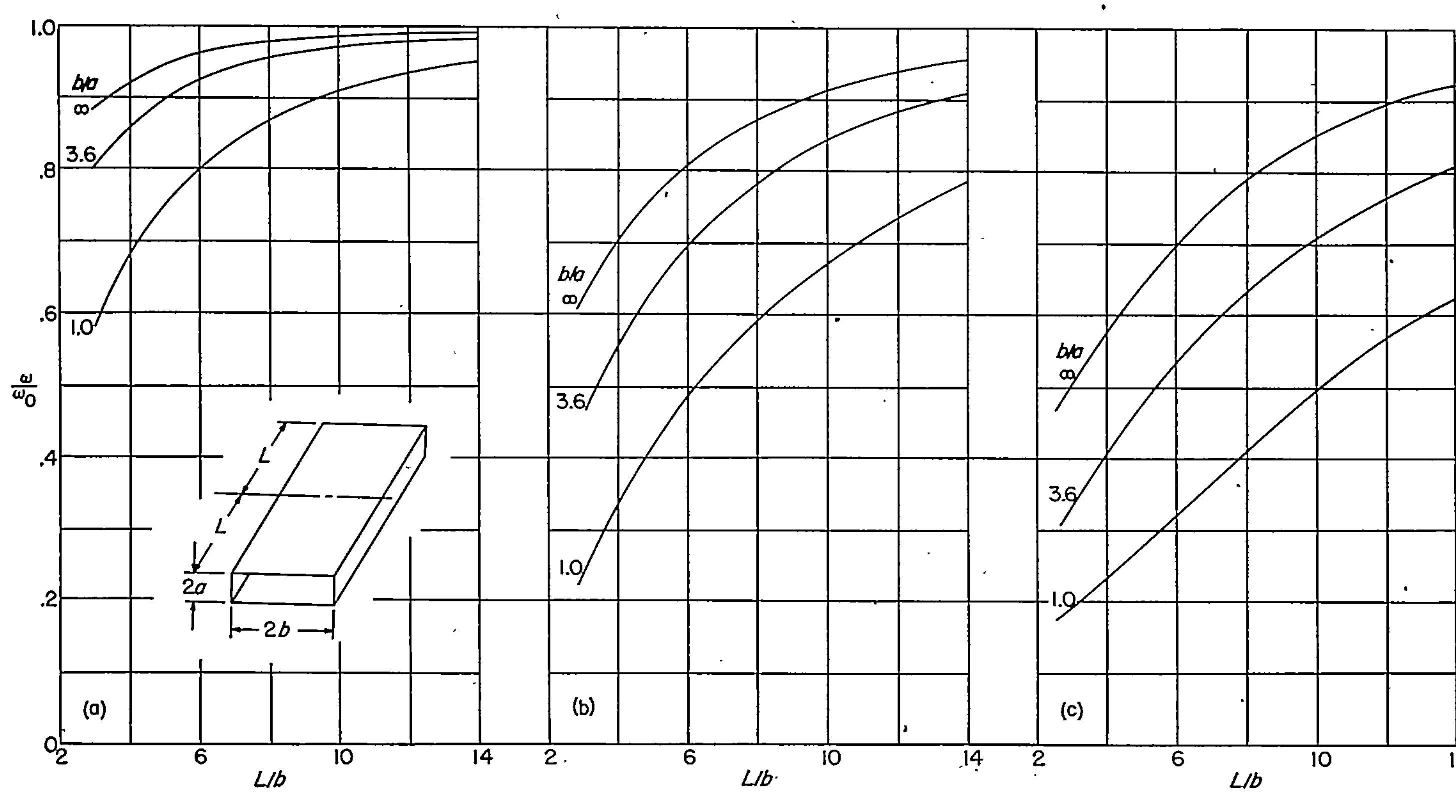
RESULTS FOR CYLINDRICAL BEAMS OF RECTANGULAR CROSS SECTION

In order to show quantitatively the effects of shear lag, transverse shear deformation, and longitudinal inertia on the natural frequencies of hollow thin-walled cylindrical beams, numerical calculations have been performed for cylinders of rectangular cross section oscillating as free-free beams. The calculations have been limited to symmetrical modes of vibration, and consequently the frequency equation (41) is applicable. For rectangular cross sections the quantity N_t may be put into closed form as shown in the appendix, and this closed-form version of N_t was used in the calculations. A value of E/G equal to 2.65 (appropriate for aluminum alloys) was assumed.

The results of these calculations are shown in figures 2, 3, and 4. In figure 2, the ratio of the natural frequency ω to the natural frequency ω_0 obtained from elementary beam theory is shown as a function of the plan-form aspect ratio L/b for cross-sectional aspect ratios of 1.0, 3.6, and ∞. The contribution of each of the secondary effects to the total reduction in the natural frequency for the cross-sectional aspect ratios $\frac{b}{a}=3.6$ and 1.0 can be seen in figures 3 and 4, respectively. The cross-sectional aspect ratio of $\frac{b}{a}=\infty$ corresponds to the limiting case of a beam where the effects of transverse shear deformation and longitudinal inertia are negligible and therefore the reduction in natural frequency is due entirely to shear lag.

The dashed lines in figures 3 and 4 show the reduction in frequency due to the inclusion of the effect of only transverse shear deformation as obtained from reference 4.

The long- and short-dash lines are calculated from the frequency equation (41) with $k_{RI}=0$ and consequently



(a) First symmetrical mode. (b) Second symmetrical mode. (c) Third symmetrical mode.

FIGURE 2.—Change in the natural frequency of a symmetrically vibrating free-free cylinder due to the inclusion of secondary effects.

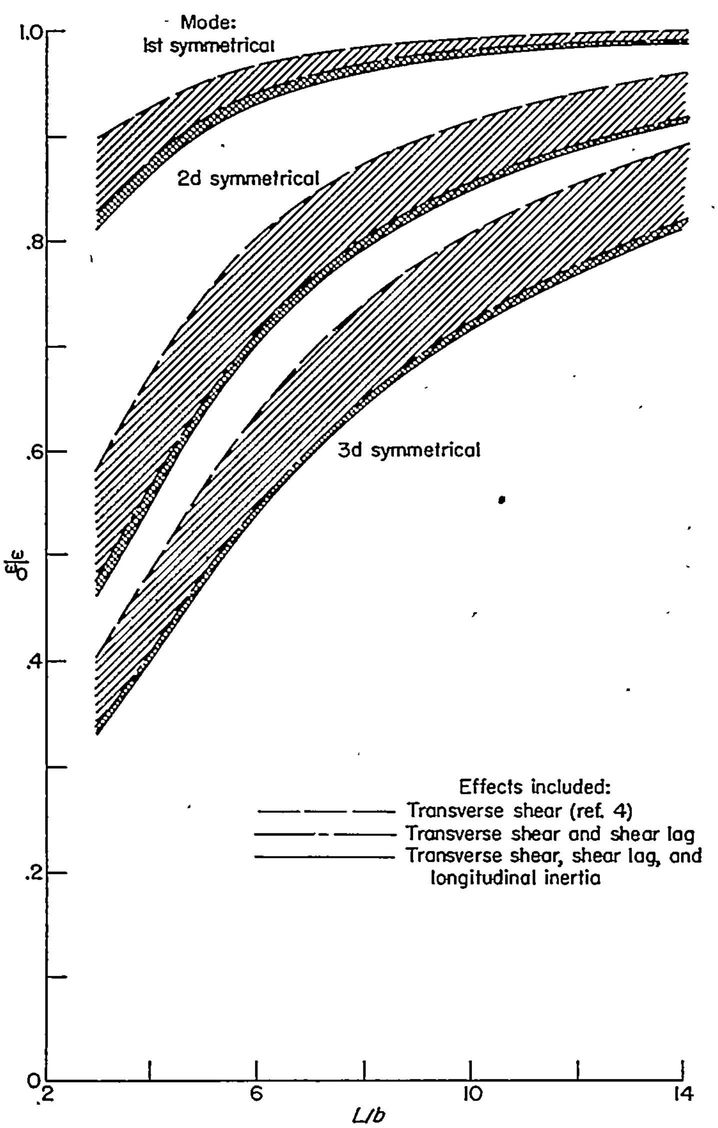


FIGURE 3.—Contribution of transverse shear deformation, shear lag, and longitudinal inertia to the reduction in natural frequency for $\frac{b}{a}=3.6$.

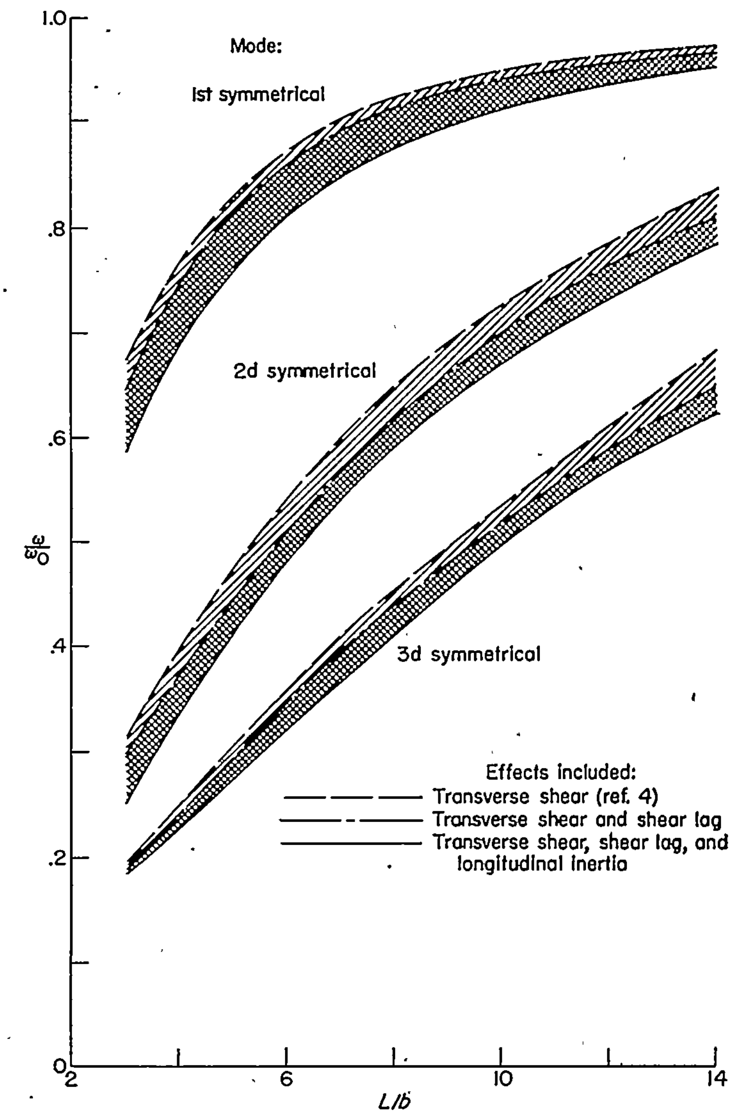


FIGURE 4.—Contribution of transverse shear deformation, shear lag, and longitudinal inertia to the reduction in natural frequency for $\frac{b}{a}=1.0$.

represent the reduction in natural frequency when both shear lag and transverse shear deformation are taken into account. Thus the hatched area between the dashed and the long- and short-dash lines may be considered as showing the additional reduction in natural frequency when the influence of shear lag is considered. Finally, the solid lines are calculated with k_{RI} taken into account, and consequently the shaded area shows the additional influence of longitudinal inertia in reducing the frequency.

Examination of figures 3 and 4 and the curves for $\frac{b}{a}=\infty$ in figure 2 shows that the influence of shear lag increases as the cross-sectional aspect ratio increases; whereas the influence of transverse shear and longitudinal inertia decreases with increasing cross-sectional aspect ratio. Indeed, it appears from the results for $\frac{b}{a}=3.6$ that for this aspect ratio the effects of longitudinal inertia may already be considered practically negligible.

A word of caution concerning the interpretation of figures 3 and 4 may be in order. Since in some cases the depth of the hatching increases with increasing L/b, it might appear, at first glance, that the shear-lag effect increases with increasing plan-form aspect ratio. However, if the additional effects

of shear lag are considered on a percentage basis with the dashed line as a base, it will be found that shear-lag effects actually reduce in percentage with increasing L/b. A similar criterion should be used in judging the influence of longitudinal inertia.

CONCLUDING REMARKS

The numerical calculations show that secondary effects have appreciable influence on the natural frequencies of rectangular box beams of uniform wall thickness. These results constitute an indication of the probable inadequacy of elementary beam theory for the vibration analysis of actual aircraft structures of the monocoque and semimonocoque type and emphasize the need for practical calculation procedures for such structures that would take into account transverse shear deformation, shear lag, and, when necessary, longitudinal inertia. The general solutions presented for cylinders of uniform thickness, as well as the numerical results for rectangular box beams, should be useful in the assessment of the accuracy of any procedure of this kind that may be developed.

Langley Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., *January 21, 1952.*

APPENDIX

TRANSFORMATION OF PARAMETERS

Expressions for A_S/A, I, and K^2.—If $\sin\theta$ is expanded into a Fourier series

$$\sin\theta=\sum_{n=1,2,3}^{\infty} A_n \sin\frac{2n\pi s}{p} \tag{A1}$$

the Fourier coefficients A_n are the same as those defined in equation (24); that is,

$$A_n=\frac{2}{p}\oint \sin\theta \sin\frac{2n\pi s}{p}\, ds \tag{A2}$$

The effective shear area A_S (eq. (25)) can now be written as a function of the Fourier series expansion for $\sin\theta$ as

$$A_S=\oint t\left(\sum_{n=1,2,3}^{\infty} A_n \sin\frac{2n\pi s}{p}\right)^2 ds \tag{A3}$$

With the use of the appropriate orthogonality conditions, equation (A3) becomes, after the integration is performed,

$$A_S=\frac{pt}{2}\sum_{n=1,2,3}^{\infty} A_n^2=\frac{A}{2}\sum_{n=1,2,3}^{\infty} A_n^2$$

or

$$\frac{A_S}{A}=\frac{1}{2}\sum_{n=1,2,3}^{\infty} A_n^2 \tag{A4}$$

The moment of inertia I of a cylinder is defined as (see fig. 1)

$$I=\int_0^p y^2 t\, ds - A\bar{y}^2 \tag{A5}$$

where $\bar{y}$ is the y-distance to the center of gravity of the cross section and is given by

$$\bar{y}=\frac{\int_0^p yt\, ds}{pt} \tag{A6}$$

But

$$y=\int_0^s \sin\theta\, ds \tag{A7}$$

or

$$y=\sum_{n=1,2,3}^{\infty} A_n \frac{p}{2n\pi}\left(1-\cos\frac{2n\pi s}{p}\right) \tag{A8}$$

and, consequently,

$$\bar{y}=\sum_{n=1,2,3}^{\infty} A_n \frac{p}{2n\pi} \tag{A9}$$

With the use of equations (A8) and (A9), the expression for I in equation (A5) becomes

$$I=\frac{Ap^2}{8\pi^2}\sum_{n=1,2,3}^{\infty} \frac{A_n^2}{n^2} \tag{A10}$$

With the series expansion for I in equation (A10), the parameter K^2, as defined in equation (29), becomes

$$K^2=\frac{2}{\pi^2}\frac{A}{A_S}\sum_{n=1,2,3}^{\infty} \frac{A_n^2}{n^2} \tag{A11}$$

Transformation of expression for N_i.—In equation (37) N_i was defined as

$$N_i=\frac{i^2\pi^2}{8k_S^2}-\frac{i^2\pi^2}{16k_S^2}K^2\frac{A}{A_S}\sum_{n=1,2,3}^{\infty}\frac{n^2A_n^2}{k_S^2B_i^2+K^2n^2}-\frac{1}{2}k_B^2 \tag{A12}$$

The infinite series that appears in this expression converges as A_n^2 and therefore is a relatively slowly converging series. In order to increase its rate of convergence, the following transformations are made.

By adding and subtracting A_n^2/K^2 inside the infinite summation in equation (A12) and using equation (A4), the equation simplifies to

$$N_i=\frac{i^2\pi^2}{16}\frac{A}{A_S}B_i^2\sum_{n=1,2,3}^{\infty}\frac{A_n^2}{k_S^2B_i^2+K^2n^2}-\frac{1}{2}k_B^2 \tag{A13}$$

By adding and subtracting A_n^2/K^2n^2 inside the infinite summation in equation (A13) and using equation (A11), the

expression for N_t can be transformed to

$$N_t=\frac{i^2\pi^4}{32}B_t^2-\frac{i^2\pi^2}{16}\frac{A}{A_S}B_t^4k_S^2\sum_{n=1,2,3}^{\infty}\frac{A_n^2}{K^2n^2(k_S^2B_t^2+K^2n^2)}-\frac{1}{2}k_B^2 \tag{A14}$$

The infinite series in equation (A14) converges as A_n^2/n^4 and therefore is considerably more quickly convergent than the series in equations (A12) and (A13), which converge as A_n^2 and A_n^2/n^2, respectively.

Closed form of N_t for cylindrical beams of rectangular cross section.—For a cylindrical beam of rectangular cross section, with dimensions as shown in figure 2, it is possible to write the expression for N_t in a closed form. The parameters for such a cross section become

$$\left.\begin{aligned} A_S&=4at \\ A&=4(a+b)t=pt \\ A_n&=0 && (n\ \text{even}) \\ &=\frac{4}{n\pi}\cos\frac{2n\pi b}{p} && (n\ \text{odd}) \end{aligned}\right\} \tag{A15}$$

With equations (A15) the parameter N_t shown in equation (A12) becomes

$$N_t=\frac{i^2\pi^2}{8k_S^2}-\frac{i^2}{4k_S^2}\frac{p}{a}\sum_{n=1,3,5}^{\infty}\frac{\cos^2\frac{2n\pi b}{p}}{\frac{k_S^2}{K^2}B_t^2+n^2}-\frac{1}{2}k_B^2 \tag{A16}$$

or

$$N_t=\frac{i^2\pi^2}{8k_S^2}-\frac{i^2}{8k_S^2}\frac{p}{a}\left(\sum_{n=1,3,5}^{\infty}\frac{1}{\frac{k_S^2}{K^2}B_t^2+n^2}+\sum_{n=1,3,5}^{\infty}\frac{\cos\frac{4n\pi b}{p}}{\frac{k_S^2}{K^2}B_t^2+n^2}\right)-\frac{1}{2}k_B^2 \tag{A17}$$

Each of the infinite summations in equation (A17) can now be written in closed form as shown in reference 5, and the closed expression for N_t then becomes

$$N_t=\frac{i^2\pi}{8k_S^2}\left\{\pi-\frac{K}{4k_SB_t}\frac{p}{a}\left[\frac{\sinh\frac{\pi}{2}\frac{k_S}{K}B_t\left(\frac{8a}{p}-1\right)}{\cosh\frac{\pi}{2}\frac{k_S}{K}B_t}+\tanh\frac{\pi}{2}\frac{k_S}{K}B_t\right]\right\}-\frac{1}{2}k_B^2 \tag{A18}$$

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